11.8 Combining the Cholesky and Square Root Transforms

The method of Section 11.7 resulted in a quasi-orthogonal matrix that was orthogonal between and within some blocks. However, there is a way to obtain a matrix that is wholly orthogonal within and between all blocks, resulting in a blockwise matrix that is strictly orthogonal.

Consider a full symmetric matrix, $\mathbf{X^T X}$, for something like $a_0 + a_1x_1 + a_2x_2 + \cdots a_{12}x_1x_2 + \cdots + a_{11}x_1^2 + a_{22}x_2^2 + \cdots$. It divides naturally into blocks per Figure 11.8-1. The terms are shown up to second order, which is almost always sufficient, although the method is effective for any $\mathbf{X^T X}$ of any order.

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
\Sigma x_0 & \Sigma x_1 & \Sigma x_2 & \cdots & \Sigma x_1x_2 & \cdots & \Sigma x_1^2 & \Sigma x_2^2 & \cdots \\
\Sigma x_1 & \Sigma x_1^2 & \Sigma x_1x_2 & \cdots & \Sigma x_1^2x_2 & \cdots & \Sigma x_1^3 & \Sigma x_1x_2^2 & \cdots \\
\Sigma x_2 & \Sigma x_1x_2 & \Sigma x_{10\ldots}^2 & \cdots & \Sigma x_1x_2^2 & \cdots & \Sigma x_1^2x_2 & \Sigma x_2^3 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\hline
\Sigma x_1x_2 & \Sigma x_1^2x_2 & \Sigma x_1x_2^2 & \cdots & \Sigma x_1^2x_2^2 & \cdots & \Sigma x_1^3x_2 & \Sigma x_1x_2^3 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\hline
\Sigma x_1^2 & \Sigma x_1^3 & \Sigma x_1^2x_2 & \cdots & \Sigma x_1^3x_2 & \cdots & \Sigma x_1^4 & \Sigma x_1^2x_2^2 & \cdots \\
\Sigma x_2^2 & \Sigma x_1x_2^2 & \Sigma x_2^3 & \cdots & \Sigma x_1x_2^3 & \cdots & \Sigma x_1^2x_2^2 & \Sigma x_2^4 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots
\end{array}\right]$$

$\mathbf{b_0}$	$\mathbf{b_1}$	$\mathbf{b_{12}}$	$\mathbf{b_{11}}$
$\mathbf{b_1}$	$\mathbf{B_{1\times1}}$	$\mathbf{B_{1\times12}}$	$\mathbf{B_{1\times11}}$
$\mathbf{b_{12}}$	$\mathbf{B_{1\times12}}$	$\mathbf{B_{12\times12}}$	$\mathbf{B_{12\times11}}$
$\mathbf{b_{11}}$	$\mathbf{B_{1\times11}}$	$\mathbf{B_{12\times11}}$	$\mathbf{B_{11\times11}}$

FIGURE 11.8-1 An arbitrary $\mathbf{X^T X}$ matrix (left panel), divided into blocks (right panel). Considering all the zero-, first-, and second-order terms with themselves and one another results in the symmetric block structure depicted in the right panel, whose blocks are generally not orthogonal.

The goal is to produce a blockwise version of the Cholesky matrix ($\mathbf{R}^T$) so that $\mathbf{RR}^T = \mathbf{LL}^T = \mathbf{X}^T\mathbf{X}$, for in that case, $\mathbf{W} = \mathbf{XR}^{-T}$ becomes orthogonal within and between all blocks. One may find $\mathbf{R}^T$ by applying the Cholesky decomposition (depicted in Figure 11.8-2, left panel) followed by square-root transforms to arrive at a blockwise rotation matrix, $\mathbf{R}^T$ (depicted in Figure 11.8-2, right panel).

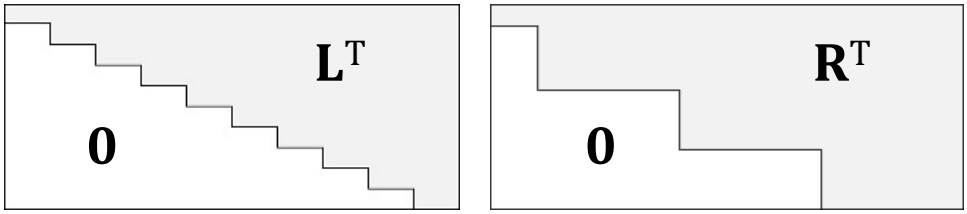

FIGURE 11.8-2 Transformation of the Cholesky decomposition ($\mathbf{L}^T$, left) to a Rotation Matrix ($\mathbf{R}^T$, right).

Here are the steps.

1. Transform $\mathbf{X^T X}$ with the transpose of its Cholesky decomposition to create an upper triangular matrix, $[Chol(\mathbf{X^T X})]^T = \mathbf{L^T}$ as depicted in the left panel of Figure 11.8-2.
2. Divide $\mathbf{L^T}$ into the desired blocks one wishes to orthogonalize and specify associated strips per Figure 11.8-3.

3. Refering to Figure 11.8-3, denote the top strip of $\mathbf{L}^{\mathrm{T}}$ as $\mathbf{p_0}$ and subsequent strips as $\mathbf{P_G}$ where $\mathbf{G}$ indexes each strip. Then build $\mathbf{R}^{\mathrm{T}}$ as stacked strips per $\mathbf{R}^{\mathrm{T}} = \begin{bmatrix} \mathbf{r_0^T} \\ \mathbf{R_1^T} \\ \mathbf{R_{12}^T} \\ \mathbf{R_{11}^T} \end{bmatrix}$, where $\mathbf{R_0^T} = \mathbf{p_0}$ (also denoted $\mathbf{r_0^T}$ as it is a row vector), and $\mathbf{R_G^T} = \left[\left(\mathbf{P_G^T P_G}\right)^{\frac{1}{2}} \mathbf{P_G^{-1}}\right] \mathbf{L_G}$ specifies all subsequent strips as depicted in Figure 11.8-4.

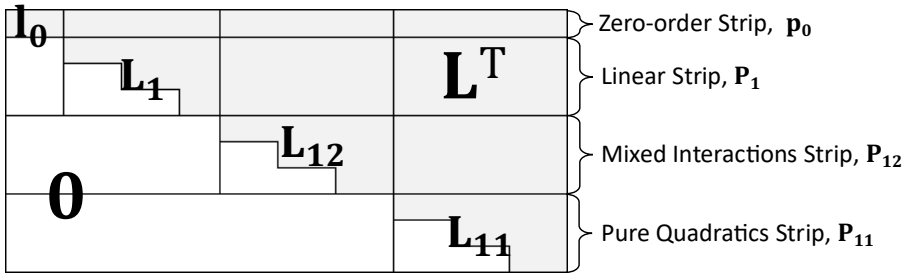

FIGURE 11.8-3 Strips and blocks in the transposed Cholesky matrix, $\mathbf{L^T}$. The figure identifies four strips ($\mathbf{p_0}$, $\mathbf{P_1}$, $\mathbf{P_{12}}$, $\mathbf{P_{11}}$). The zero-order associated with three square blocks ($\mathbf{L_1}$, $\mathbf{L_{12}}$, $\mathbf{L_{11}}$) one desires to transform from $\mathbf{L^T}$.

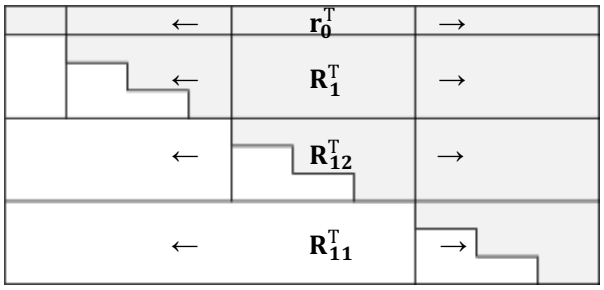

FIGURE 11.8-4 The rotation matrix as stacked strips; the R^{T} matrix comprises the stacked strips $\mathbf{r_0^T}, \mathbf{R_1^T}, \mathbf{R_{12}^T}, \mathbf{R_{11}^T}$.

4. Multiply $\mathbf{X}$ by $\mathbf{R^T}$ to create $\mathbf{W} = \mathbf{X}\mathbf{R}^{-\mathrm{T}}$, a matrix such that each block is orthogonal within blocks and between all blocks. The resulting matrix is strictly orthogonal; i.e., $\mathbf{W^T W} = \mathbf{I}$.

Example 11.8-1: Blockwise Orthogonalization of X.

Problem Statement:

Given: the data of Example 11.6-1, do the following.

Find: 1. the Cholesky decomposition of $\mathbf{X^T X}$;

2. derive a rotation matrix $\mathbf{R^T}$ from $\mathbf{L^T}$ and apply it to orthogonalize $\mathbf{X}$ in zero-, first-, mixed- and second-order blocks;

3. show the resulting $\mathbf{W}$ matrix;

4. show that $\mathbf{W} = \mathbf{X}\mathbf{R}^{-\mathrm{T}}$, $\mathbf{U} = \mathbf{X}\mathbf{S}^{-1}$ and $\mathbf{V} = \mathbf{X}\mathbf{L}^{-\mathrm{T}}$ are all orthogonal even though $\mathbf{R}^{\mathrm{T}} \neq \mathbf{L}^{\mathbf{T}} \neq \mathbf{S}$;
5. derive the coefficients for each matrix and show that they lead to the same result; i.e., $\hat{\mathbf{y}} = \mathbf{X}\mathbf{a} = \mathbf{U}\mathbf{b} = \mathbf{W}\mathbf{c} = \mathbf{V}\mathbf{d}$;
6. produce the ANOVA for each component and for the groups. Show that they sum to the same sums of squares for the overall model. Which blocks and components are significant at the 95% confidence level?

Solution: Refer to the spreadsheet depicted in Figure 18-5.

1. =Chol(XTX) gives the Cholesky decomposition* of $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ in Cells N13:X24.
2. From them, one derives the rotation matrix, $\mathbf{R}^{\mathrm{T}}$, strip by strip from top to bottom as first row: p_0; for the next three rows =MMULT(MMULT(MSQRT(MMULT(TRANSPOSE(L_1),L_1)),MINVERSE(L_1)),P_1); for the second set of three rows; =MMULT(MMULT(MSQRT(MMULT(TRANSPOSE(L_12),L_12)), MINVERSE(L_12)),P_12) for the final set of three rows; and =MMULT(MMULT(MSQRT(MMULT(TRANSPOSE(L_11),L_11)),MINVERSE(L_11)),P_11); thus building $\mathbf{R}^{\mathrm{T}}$ in Cells N25:X36.
3. $\mathbf{W} = \mathbf{X}\mathbf{R}^{-\mathrm{T}}$ derives $\mathbf{W}$ in A19:K36. Cells N49:X60 show $\mathbf{W}^{\mathbf{T}}\mathbf{W} = \mathbf{I}$, proving that $\mathbf{W}$ is is orthogonal in the strictest sense.
4. From inspection, $\mathbf{W} = \mathbf{X}\mathbf{R}^{-\mathrm{T}}$, $\mathbf{U} = \mathbf{X}\mathbf{S}^{-1}$, and $\mathbf{V} = \mathbf{X}\mathbf{L}^{-\mathrm{T}}$ differ from one another and from $\mathbf{R}^{\mathrm{T}}$ (compare Cells A19:K35, N13:X24, N25:X36, and N37:X48). Yet $\mathbf{W}^{\mathbf{T}}\mathbf{W} = \mathbf{U}^{\mathbf{T}}\mathbf{U} = \mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{I}$; thus, all are strictly orthogonal†.
5. All these matrices generate identical predictions with their respective coefficients; that is, $\hat{\mathbf{y}} = \mathbf{X}\mathbf{a} = \mathbf{U}\mathbf{b} = \mathbf{W}\mathbf{c} = \mathbf{V}\mathbf{d}$, as show in Columns L:M of the spreadsheet (with L3:L18 holding the original y values which form the basis of all the regressions). One derives the respective coefficients directly from their matrices ($\mathbf{b} = \mathbf{U}^{\mathrm{T}}\mathbf{y}$, $\mathbf{c} = \mathbf{W}^{\mathrm{T}}\mathbf{y}$, $\mathbf{d} = \mathbf{V}^{\mathrm{T}}\mathbf{y}$) or from the $\mathbf{a}$ coefficients; i.e., $\mathbf{b} = \mathbf{S}\mathbf{a}$, $\mathbf{c} = \mathbf{R}^{\mathrm{T}}\mathbf{a}$, $\mathbf{d} = \mathbf{L}^{\mathrm{T}}\mathbf{a}$, as shown in Columns Z:AA.
6. Columns AB:AH hold the ANOVAs. The ANOVA for the individual components of each block is in AB1:AH15. Note that the sum of squares for each component is simply that coefficient squared. Dividing each sum of squares by the total sum of squares gives individual coefficients of determination (AH4:AH12), each being a direct assessment of practical significance for the respective effect. Cells AB16:AH24 hold the ANOVA for the grouped components. Note that the sum of squares for each group is identical to the sum of squares of its components. Of necessity, both ANOVAs for the components and groups give exactly the same apportionments of variance for SSM, SSR, and SST. Per their respective F tests, all groups are significant at the 95% confidence level. However, the component-wise ANOVA shows that the w_{22} component in the W_{11} group is not statistically significant. It also has virtually no practical significance; therefore, omitting it from the model would be inconsequential.

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* The Chol() and MSQRT() commands require the Real-Statistics add-in for Excel, last accessed 22 November 2026: https://real-statistics.com/free-download/.

† One could also show that $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{L}\mathbf{L}^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}}\mathbf{X}$; thus, $\mathbf{R}\mathbf{R}^{\mathrm{T}}$ is an alternate (blockwise) decomposition to the Cholesky.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
		1.) X Matrix										y	ŷ =		2.) X^TX Matrix									
1	Pt	0	1	2	3	12	13	23	11	22	33		Xa	Pt	0	1	2	3	12	13	23	11	22	33
2	1	1	-0.761	-0.781	0.578	0.594	-0.440	-0.452	0.579	0.611	0.334	-0.121	-0.169	0	16.000	1.441	-1.721	1.942	0.196	2.359	-0.889	4.333	6.024	7.414
3	2	1	-0.479	0.162	-0.951	-0.078	0.456	-0.154	0.230	0.026	0.905	-0.550	-0.398	1	1.441	4.333	0.196	2.359	-0.871	1.623	-0.317	1.484	0.403	0.339
4	3	1	-0.415	0.805	-0.824	-0.334	0.342	-0.663	0.172	0.648	0.680	-1.442	-1.564	2	-1.721	0.196	6.024	-0.889	0.403	-0.317	-0.110	-0.871	-1.592	-1.305
5	4	1	-0.394	-0.321	0.878	0.126	-0.346	-0.281	0.155	0.103	0.771	-1.182	-1.077	3	1.942	2.359	-0.889	7.414	-0.317	0.339	-1.305	1.623	-0.110	0.560
6	5	1	-0.295	-0.999	-0.759	0.295	0.224	0.758	0.087	0.998	0.576	1.516	1.463	12	0.196	-0.871	0.403	-0.317	1.699	-1.065	0.855	-0.118	0.264	-0.345
7	6	1	-0.291	0.083	0.357	-0.024	-0.104	0.030	0.085	0.007	0.128	-0.420	-0.536	13	2.359	1.623	-0.317	0.339	-1.065	2.043	-0.345	1.178	0.855	1.626
8	7	1	-0.232	-0.684	-0.669	0.159	0.156	0.458	0.054	0.468	0.448	1.077	1.087	23	-0.889	-0.317	-0.110	-1.305	0.855	-0.345	2.493	-1.065	-0.592	-0.494
9	8	1	-0.109	0.625	0.153	-0.068	-0.017	0.095	0.012	0.391	0.023	-1.106	-0.937	11	4.333	1.484	-0.871	1.623	-0.118	1.178	-1.065	2.257	1.699	2.043
10	9	1	0.219	0.235	0.594	0.051	0.130	0.139	0.048	0.055	0.353	-0.951	-1.126	22	6.024	0.403	-1.592	-0.110	0.264	0.855	-0.592	1.699	4.096	2.493
11	10	1	0.273	-0.316	-0.751	-0.086	-0.205	0.237	0.074	0.100	0.563	0.351	0.318	33	7.414	0.339	-1.305	0.560	-0.345	1.626	-0.494	2.043	2.493	4.600
12	11	1	0.350	0.422	0.914	0.147	0.320	0.386	0.122	0.178	0.836	-1.967	-1.954	3.) L^T Matrix										
13	12	1	0.423	-0.990	0.537	-0.419	0.227	-0.532	0.179	0.981	0.289	-1.220	-1.089		0	1	2	3	12	13	23	11	22	33
14	13	1	0.665	0.907	-0.177	0.603	-0.118	-0.161	0.442	0.823	0.031	-1.286	-1.303	0	4.000	0.360	-0.430	0.486	0.049	0.590	-0.222	1.083	1.506	1.854
15	14	1	0.715	-0.362	0.700	-0.259	0.501	-0.254	0.512	0.131	0.491	-1.917	-2.059	1		2.050	0.171	1.066	-0.434	0.688	-0.116	0.534	-0.068	-0.160
16	15	1	0.813	0.181	0.507	0.148	0.412	0.092	0.661	0.033	0.257	-2.042	-1.888	2			2.410	-0.358	0.207	-0.075	-0.077	-0.206	-0.387	-0.199
17	16	1	0.960	-0.687	0.854	-0.660	0.820	-0.587	0.922	0.473	0.729	-3.323	-3.351	3				2.432	0.080	-0.291	-0.453	0.187	-0.373	-0.099
18	5.) $W = XR^{-T}$											y	ŷ =	12					1.208	-0.626	0.718	0.073	0.224	-0.377
19	Pt	0	1	2	3	12	13	23	11	22	33		Wc	13						0.860	0.208	0.298	0.019	0.422
20	1	0.250	-0.474	-0.246	0.246	0.455	-0.086	-0.323	0.236	-0.036	-0.086	-0.121	-0.169	23							1.288	-0.627	-0.488	0.018
21	2	0.250	-0.203	0.098	-0.365	-0.020	0.404	-0.154	-0.151	-0.438	0.115	-0.550	-0.398	11								0.484	-0.483	0.024
22	3	0.250	-0.199	0.368	-0.303	-0.278	0.205	-0.415	0.011	0.259	0.108	-1.442	-1.564	22									1.006	-0.324
23	4	0.250	-0.322	-0.055	0.343	-0.068	-0.245	-0.047	-0.044	-0.088	0.451	-1.182	-1.077	33										0.814
24	5	0.250	-0.094	-0.386	-0.338	0.215	0.181	0.382	0.111	0.412	0.027	1.516	1.463	3.) R^T Matrix										
25	6	0.250	-0.228	0.096	0.136	-0.223	-0.128	0.154	0.131	-0.170	-0.395	-0.420	-0.536		0	1	2	3	12	13	23	11	22	33
26	7	0.250	-0.078	-0.254	-0.300	0.087	0.059	0.222	-0.148	-0.097	-0.157	1.077	1.087	0	4.000	0.360	-0.430	0.486	0.049	0.590	-0.222	1.083	1.506	1.854
27	8	0.250	-0.127	0.312	0.052	-0.276	-0.127	0.201	0.249	0.287	-0.379	-1.106	-0.937	1		1.992	0.095	0.475	-0.446	0.739	-0.004	0.481	0.033	-0.127
28	9	0.250	0.013	0.153	0.185	-0.042	0.003	0.215	-0.343	-0.149	-0.104	-0.951	-1.126	2		0.095	2.410	-0.144	0.191	-0.063	-0.112	-0.168	-0.414	-0.212
29	10	0.250	0.186	-0.116	-0.371	-0.229	-0.611	0.101	0.193	-0.246	0.321	0.351	0.318	3		0.475	-0.144	2.633	-0.039	-0.117	-0.460	0.321	-0.348	-0.118
30	11	0.250	0.046	0.236	0.306	0.060	0.235	0.416	-0.124	0.201	0.472	-1.967	-1.954	12					1.109	-0.348	0.330	0.332	0.356	-0.241
31	12	0.250	0.158	-0.366	0.109	-0.219	-0.141	-0.278	-0.560	0.303	-0.189	-1.220	-1.089	13					-0.348	1.003	-0.064	0.291	-0.026	0.512
32	13	0.250	0.304	0.400	-0.146	0.553	-0.303	-0.226	-0.099	0.218	-0.010	-1.286	-1.303	23					0.330	-0.064	1.451	-0.541	-0.402	-0.033
33	14	0.250	0.280	-0.107	0.164	-0.041	0.139	-0.077	-0.161	-0.269	-0.085	-1.917	-2.059	11								0.460	-0.152	-0.011
34	15	0.250	0.336	0.112	0.092	0.255	0.129	0.088	0.206	-0.306	-0.211	-2.042	-1.888	22								-0.152	1.092	-0.174
35	16	0.250	0.403	-0.245	0.192	-0.229	0.285	-0.259	0.493	0.118	0.122	-3.323	-3.351	33								-0.011	-0.174	0.859

FIGURE 11.8-5 Part 1: the spreadsheet. This section of the spreadsheet shows 1.) The starting X matrix. 2.) The subsequent X^TX Matrix. The transposed Cholesky Matrix. 4.) The rotation matrix, R. The W Matrix.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
36	**7.) U = XS^{-1}**											**y**	**ŷ =**	**6.) S = √(X^{T}X)**										
37	**Pt**	**0**	**1**	**2**	**3**	**12**	**13**	**23**	**11**	**22**	**33**		**Ub**		**0**	**1**	**2**	**3**	**12**	**13**	**23**	**11**	**22**	**33**
38	**1**	0.164	-0.431	-0.267	0.199	0.390	-0.238	-0.247	0.419	0.147	-0.023	-0.121	-0.169	**0**	3.517	0.172	-0.151	0.280	0.076	0.308	-0.059	0.735	1.045	1.327
39	**2**	0.246	-0.228	0.042	-0.385	0.134	0.385	-0.216	0.015	-0.284	0.326	-0.550	-0.398	**1**	0.172	1.903	0.102	0.494	-0.252	0.451	0.030	0.388	0.066	-0.065
40	**3**	0.135	-0.247	0.407	-0.285	-0.273	0.130	-0.355	0.028	0.309	0.246	-1.442	-1.564	**2**	-0.151	0.102	2.392	-0.182	0.144	-0.019	-0.098	-0.184	-0.346	-0.232
41	**4**	0.158	-0.167	-0.075	0.327	0.041	-0.392	-0.143	-0.080	-0.074	0.462	-1.182	-1.077	**3**	0.280	0.494	-0.182	2.617	-0.026	-0.019	-0.300	0.308	-0.149	0.025
42	**5**	0.074	-0.144	-0.315	-0.212	0.094	0.139	0.529	0.069	0.487	0.150	1.516	1.463	**12**	0.076	-0.252	0.144	-0.026	1.116	-0.433	0.362	0.109	0.136	-0.115
43	**6**	0.448	-0.248	0.048	0.136	-0.179	-0.073	0.093	0.029	-0.177	-0.270	-0.420	-0.536	**13**	0.308	0.451	-0.019	-0.019	-0.433	1.136	-0.009	0.309	0.131	0.392
44	**7**	0.301	-0.070	-0.288	-0.255	0.139	0.159	0.201	-0.104	0.030	-0.014	1.077	1.087	**23**	-0.059	0.030	-0.098	-0.300	0.362	-0.009	1.441	-0.379	-0.177	-0.070
45	**8**	0.355	-0.190	0.336	0.111	-0.343	-0.096	0.241	0.050	0.206	-0.286	-1.106	-0.937	**11**	0.735	0.388	-0.184	0.308	0.109	0.309	-0.379	1.008	0.255	0.326
46	**9**	0.353	0.068	0.089	0.217	0.114	0.135	0.041	-0.292	-0.127	-0.001	-0.951	-1.126	**22**	1.045	0.066	-0.346	-0.149	0.136	0.131	-0.177	0.255	1.626	0.283
47	**10**	0.298	0.340	-0.142	-0.401	-0.229	-0.571	0.129	0.031	-0.183	0.267	0.351	0.318	**33**	1.327	-0.065	-0.232	0.025	-0.115	0.392	-0.070	0.326	0.283	1.557
48	**11**	0.044	0.080	0.273	0.409	0.118	0.138	0.345	-0.146	0.120	0.535	-1.967	-1.954	**9.) W^{T}W = R^{-T}(X^{T}X)R^{-1} = I**										
49	**12**	0.239	0.218	-0.386	0.141	-0.150	0.088	-0.387	-0.422	0.420	-0.107	-1.220	-1.089		**0**	**1**	**2**	**3**	**12**	**13**	**23**	**11**	**22**	**33**
50	**13**	0.182	0.417	0.384	-0.147	0.480	-0.186	-0.160	0.133	0.391	-0.082	-1.286	-1.303	**0**	1									
51	**14**	0.258	0.263	-0.159	0.150	0.011	0.270	-0.162	0.032	-0.166	0.030	-1.917	-2.059	**1**		1								
52	**15**	0.274	0.273	0.071	0.078	0.188	0.217	0.116	0.390	-0.204	-0.127	-2.042	-1.888	**2**			1							
53	**16**	-0.010	0.240	-0.170	0.195	-0.457	0.203	-0.085	0.585	0.151	0.220	-3.323	-3.351	**3**				1						
54	**8.) V= XL^{-T}**											**y**	**ŷ =**	**12**					1					
55	**Pt**	**0**	**1**	**2**	**3**	**12**	**13**	**23**	**11**	**22**	**33**		**Vd**	**13**						1				
56	**1**	0.250	-0.415	-0.250	0.333	0.354	-0.002	-0.440	0.237	0.054	-0.072	-0.121	-0.169	**23**							1			
57	**2**	0.250	-0.278	0.132	-0.300	-0.177	0.362	-0.159	-0.009	-0.476	0.027	-0.550	-0.398	**11**								1		
58	**3**	0.250	-0.246	0.396	-0.223	-0.428	0.071	-0.321	-0.073	0.226	0.151	-1.442	-1.564	**22**									1	
59	**4**	0.250	-0.236	-0.072	0.404	-0.005	-0.258	-0.015	-0.025	-0.179	0.425	-1.182	-1.077	**33**										1
60	**5**	0.250	-0.188	-0.356	-0.332	0.249	0.277	0.293	-0.025	0.414	0.105	1.516	1.463	**10.) U^{T}U = I**										
61	**6**	0.250	-0.186	0.092	0.192	-0.125	-0.162	0.219	0.187	-0.046	-0.406	-0.420	-0.536		**0**	**1**	**2**	**3**	**12**	**13**	**23**	**11**	**22**	**33**
62	**7**	0.250	-0.157	-0.228	-0.290	0.123	0.107	0.183	-0.107	-0.107	-0.183	1.077	1.087	**0**	1									
63	**8**	0.250	-0.097	0.311	0.101	-0.161	-0.169	0.280	0.156	0.414	-0.304	-1.106	-0.937	**1**		1								
64	**9**	0.250	0.063	0.138	0.187	0.019	0.019	0.217	-0.276	-0.224	-0.154	-0.951	-1.126	**2**			1							
65	**10**	0.250	0.089	-0.093	-0.411	-0.006	-0.633	0.191	0.253	-0.230	0.290	0.351	0.318	**3**				1						
66	**11**	0.250	0.127	0.211	0.301	0.101	0.293	0.369	-0.192	0.063	0.488	-1.967	-1.954	**12**					1					
67	**12**	0.250	0.162	-0.378	0.044	-0.237	-0.228	-0.194	-0.622	0.147	-0.180	-1.220	-1.089	**13**						1				
68	**13**	0.250	0.280	0.401	-0.187	0.533	-0.172	-0.367	-0.163	0.176	0.019	-1.286	-1.303	**23**							1			
69	**14**	0.250	0.305	-0.127	0.086	-0.099	0.113	-0.067	-0.067	-0.285	-0.142	-1.917	-2.059	**11**								1		
70	**15**	0.250	0.353	0.095	0.018	0.221	0.201	0.003	0.297	-0.184	-0.242	-2.042	-1.888	**22**									1	
71	**16**	0.250	0.424	-0.271	0.076	-0.363	0.180	-0.191	0.428	0.238	0.179	-3.323	-3.351	**33**										1

FIGURE 11.8-5 Part 2: the spreadsheet (continued). This section of the spreadsheet shows 6.) S, the square root of the X^TX matrix. 7.) The resulting $U = XS^{-1}$ matrix. 8.) V, the Cholesky transform of the X matrix. 9.) Proof that the blockwise W matrix is orthogonal. 10. Proof that the U matrix is orthogonal.

	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH	AI
		Coeff		Term	ANOVA, Component Model						
1	Pt	a			SS	DF	MS	F	p	R_a^2	stat sig
2	0	-0.051		w_0	13.294	—	—	—	—	—	—
3	1	-0.365		w_1	5.458	1	5.458	172.24	0.0000	24.5%	*
4	2	-0.996		w_2	2.763	1	2.763	87.20	0.0001	12.4%	*
5	3	-0.865		w_3	7.512	1	7.512	237.06	0.0000	33.7%	*
6	12	1.016		w_{12}	0.941	1	0.941	29.68	0.0016	4.2%	*
7	13	-0.160		w_{13}	1.644	1	1.644	51.89	0.0004	7.4%	*
8	23	0.415		w_{23}	3.163	1	3.163	99.83	0.0001	14.2%	*
9	11	-1.278		w_{11}	0.286	1	0.286	9.04	0.0238	1.3%	*
10	22	-0.288		w_{22}	0.000	1	0.000	0.00	0.9657	0.0%	
11	33	-0.737		w_{33}	0.323	1	0.323	10.21	0.0187	1.5%	*
12		Coeff		M	22.091	9	2.455	77.46	0.0000		
13	Pt	$d=L^Ta=V^Ty$		R	0.190	6	0.032	s =	0.178		
14	0	-3.646	-3.646	T	22.281	15		R_2 =	99.15%	99.15%	
15	1	-2.983	-2.983	Term	ANOVA, Group Model						
16	2	-1.379	-1.379		SS	DF	MS	F	p	R_a^2	$e_j=\Sigma b_{jk}$
17	3	-2.221	-2.221	w_0	1.159	—	—	—	—	—	-3.646
18	12	1.746	1.746	W_1	15.733	3	5.244	165.50	0.0000	70.6%	3.966
19	13	-0.749	-0.749	W_{12}	5.748	3	1.916	60.47	0.0001	25.8%	2.398
20	23	1.463	1.463	W_{11}	0.610	3	0.203	6.42	0.0266	2.7%	0.781
21	11	-0.497	-0.497	M	22.091	9	2.455	77.46	0.0000		
22	22	-0.051	-0.051	R	0.190	6	0.032	s =	0.178		
23	33	-0.600	-0.600	T	22.281	15		R_2 =	99.15%	99.15%	
24		Coeff		Group Model, G							
25	Pt	$c=R^Ta=W^Ty$		Pt	g_0	g_1	g_{12}	g_{11}	y	$\hat{y}=Ge$	
26	0	-3.646	-3.646	1	0.250	0.213	-0.010	-0.099	-0.121	-0.169	
27	1	-2.336	-2.336	2	0.250	0.331	-0.339	0.016	-0.550	-0.398	
28	2	-1.662	-1.662	3	0.250	0.172	-0.530	-0.084	-1.442	-1.564	
29	3	-2.741	-2.741	4	0.250	-0.024	0.068	-0.299	-1.182	-1.077	
30	12	0.970	0.970	5	0.250	0.451	0.274	-0.092	1.516	1.463	
31	13	-1.282	-1.282	6	0.250	0.000	0.093	0.196	-0.420	-0.536	
32	23	1.779	1.779	7	0.250	0.360	0.168	0.215	1.077	1.087	
33	11	-0.535	-0.535	8	0.250	-0.092	0.106	0.108	-1.106	-0.937	
34	22	0.008	0.008	9	0.250	-0.200	0.141	0.309	-0.951	-1.126	
35	33	-0.569	-0.569	10	0.250	0.196	0.309	-0.369	0.351	0.318	
36	Y	Z	AA	11	0.250	-0.337	0.207	-0.257	-1.967	-1.954	
37		Coeff		12	0.250	-0.015	-0.219	0.525	-1.220	-1.089	
38	Pt	$b=Sa=U^Ty$		13	0.250	-0.246	0.218	0.077	-1.286	-1.303	
39	0	-2.547	-2.547	14	0.250	-0.233	-0.148	0.170	-1.917	-2.059	
40	1	-2.013	-2.013	15	0.250	-0.308	0.100	0.009	-2.042	-1.888	
41	2	-1.639	-1.639	16	0.250	-0.267	-0.437	-0.425	-3.323	-3.351	
42	3	-2.794	-2.794								
43	12	1.228	1.228								
44	13	-1.491	-1.491								
45	23	1.903	1.903								
46	11	-1.958	-1.958								
47	22	-0.564	-0.564								
48	33	-1.687	-1.687								

49	11.) $V^TV = I$										
50		0	1.000	2	3	12	13	23	11	22	33
51	0	1									
52	1		1								
53	2			1							
54	3				1						
55	12					1					
56	13						1				
57	23							1			
58	11								1		
59	22									1	
60	33										1

FIGURE 11.8-5 Part 3: the spreadsheet continued. Columns Z:AA give the coefficients of the various models with the blockwise coefficients in Z27:AA36. Y49:AI60 proves that V is strictly orthogonal. AB1:AI14 gives the component wise ANOVA. AB16:AI24 gives the groupwise ANOVA. AB25:AH42 shows the groupwise contributions and regressed y values.