## 11.8 The Blockwise Cholesky Decomposition

The method of Section 11.7 resulted in a quasi-orthogonal matrix that was orthogonal between blocks and within some others. However, there is a way to obtain a matrix that is orthogonal within and between all blocks, resulting in a blockwise matrix that is strictly orthogonal.

Consider a full symmetric matrix, $\mathbf{X^T X}$, for something like $a_0 + a_1x_1 + a_2x_2 + \cdots a_{12}x_1x_2 + \cdots + a_{11}x_1^2 + a_{22}x_2^2 + \cdots$ . It divides naturally into blocks per Figure 11.8-1. The terms are shown up to second order, which is almost always sufficient, although the method is effective for any $\mathbf{X^T X}$ of any order.

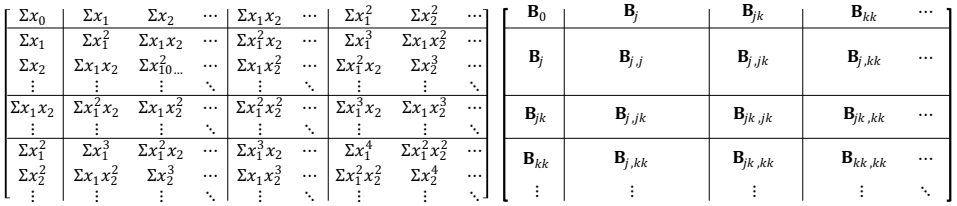

**FIGURE 11.8-1** An arbitrary $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ matrix (left panel), divided into blocks (right panel). Considering all the zero-, first-, and second-order terms with themselves and one another results in the symmetric block structure depicted in the right panel, whose blocks are generally not orthogonal.

The goal is to produce a blockwise version of the Cholesky matrix ($\mathbf{R}^{\mathrm{T}}$) so that $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{L}\mathbf{L}^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}}\mathbf{X}$, for in that case, $\mathbf{W} = \mathbf{X}\mathbf{R}^{-\mathrm{T}}$ becomes orthogonal within and between all blocks. One may find $\mathbf{R}^{\mathrm{T}}$ by applying the Cholesky decomposition (depicted in Figure 11.8-2, left panel) followed by square-root transforms to arrive at a blockwise rotation matrix, $\mathbf{R}^{\mathrm{T}}$ (depicted in Figure 11.8-2, right panel).

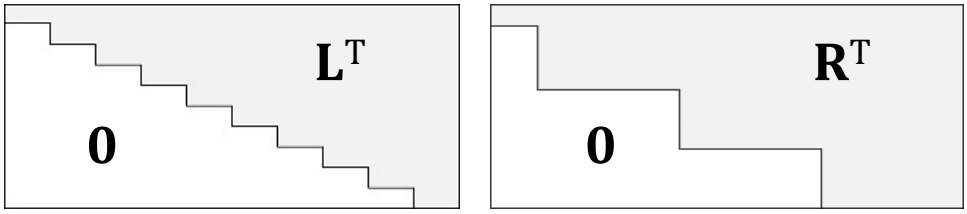


**FIGURE 11.8-2** Transformation of the Cholesky decomposition ($\mathbf{L}^{\mathrm{T}}$, left) to a Rotation Matrix ($\mathbf{R}^{\mathrm{T}}$, right).

Here are the steps.

1. Transform $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ with the transpose of its Cholesky decomposition to create an upper triangular matrix, $[Chol(\mathbf{X}^{\mathrm{T}}\mathbf{X})]^{\mathrm{T}} = \mathbf{L}^{\mathbf{T}}$ as depicted in the left panel of Figure 11.8-2.
2. Divide $\mathbf{L}^{\mathbf{T}}$ into the desired blocks one wishes to orthogonalize and specify associated strips per Figure 11.8-3.

3. Refering to Figure 11.8-3, let $\mathbf{R}_G^{\mathrm{T}} = \left[(\mathbf{L}_G^{\mathbf{T}}\mathbf{L}_G)^{\frac{1}{2}}\mathbf{L}_G^{-1}\right]\mathbf{P}_G$, where $G$ indexes each strip; e.g. $G = \{0, j, jk, kk\}$*. Then build $\mathbf{R}^{\mathrm{T}}$ as stacked strips per $\mathbf{R}^{\mathrm{T}} = \begin{bmatrix}\mathbf{R}_{\mathbf{0}}^{\mathrm{T}}\\ \mathbf{R}_j^{\mathrm{T}}\\ \mathbf{R}_{jk}^{\mathrm{T}}\\ \mathbf{R}_{kk}^{\mathrm{T}}\end{bmatrix}$, noting that $\mathbf{R}_{\mathbf{0}}^{\mathrm{T}}$ simplifies to $\mathbf{R}_0^{\mathrm{T}} = \mathbf{r}_0^{\mathrm{T}} = \mathbf{p}_0$ (see Figure 11.8-4 for a depiction).

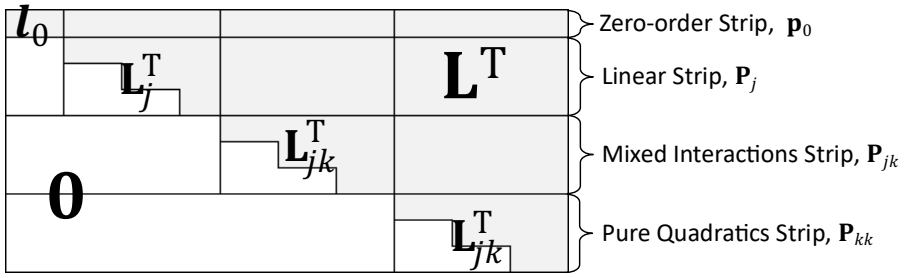


**FIGURE 11.8-3** Strips and blocks in the transposed Cholesky matrix, $\mathbf{L}^{\mathbf{T}}$. The figure identifies four strips ($\mathbf{p}_0$, $\mathbf{P}_j, \mathbf{P}_{jk}, \mathbf{P}_{kk}$), each respectively containing the square blocks ($\mathbf{l}_0$, $\mathbf{L}_j$, $\mathbf{L}_{jk}$, $\mathbf{L}_{kk}$) one desires to transform from $\mathbf{L}^{\mathbf{T}}$.

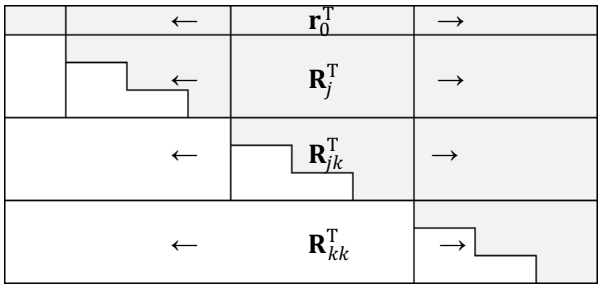


**FIGURE 11.8-4** The rotation matrix as stacked strips; the $\mathbf{R}^{\mathrm{T}}$ matrix comprises the stacked strips $\mathbf{r}_{\mathbf{0}}^{\mathrm{T}}, \mathbf{R}_j^{\mathrm{T}}, \mathbf{R}_{jk}^{\mathrm{T}}, \mathbf{R}_{kk}^{\mathrm{T}}$.

4. Multiply $\mathbf{X}$ by $\mathbf{R}^{\mathrm{T}}$ to create $\mathbf{W} = \mathbf{X}\mathbf{R}^{-\mathrm{T}}$, a matrix such that each block is orthogonal within blocks and between all blocks. The resulting matrix is strictly orthogonal; i.e., $\mathbf{W}^{\mathbf{T}}\mathbf{W} = \mathbf{I}$.

### Example 11.8-1: Blockwise Orthogonalization of X.

**Problem Statement:** Given: the data of Example 11.6-1, do the following. Find: 1.) the Cholesky decomposition of $\mathbf{X}^{\mathrm{T}}\mathbf{X}$; 2.) derive a rotation matrix $\mathbf{R}^{\mathrm{T}}$ from $\mathbf{L}^{\mathrm{T}}$ and apply it to

* One may also define extraction matrices, $\mathbf{E}_G$, corresponding to the grouped columns of the identity matrix. Then, $\mathbf{P}_G = \mathbf{E}_G\mathbf{L}^{\mathrm{T}}$, leading to $\mathbf{R}_G^{\mathrm{T}} = \left[\left(\mathbf{L}_G^{\mathbf{T}}\mathbf{L}_G\right)^{\frac{1}{2}}\mathbf{L}_G^{-1}\right]\mathbf{E}_G^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}$. Note also that $\mathbf{L}_G = \mathbf{E}_G^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{E}_G$.

orthogonalize $\mathbf{X}$ in zero-, first-, mixed- and second-order blocks; 3.) show the resulting $\mathbf{W}$ matrix; 4.) show that $\mathbf{W} = \mathbf{XR}^{-\mathrm{T}}$, $\mathbf{U} = \mathbf{XS}^{-1}$ and $\mathbf{V} = \mathbf{XL}^{-\mathrm{T}}$ are all orthogonal even though $\mathbf{R}^{\mathrm{T}} \neq \mathbf{L}^{\mathbf{T}} \neq \mathbf{S}$; 5.) derive the coefficients for each matrix and show that they lead to the same result; i.e., $\hat{\mathbf{y}} = \mathbf{Xa} = \mathbf{Ub} = \mathbf{Wc} = \mathbf{Vd}$; 6.) produce the ANOVA for each component and for the groups. Show that they sum to the same sums of squares for the overall model. Which blocks and components are significant at the 95% confidence level?

### *Solution: Refer to the spreadsheet depicted in Figure 18-5.*

1. =Chol(XTX) gives the Cholesky decomposition† of $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ in Cells N13:X24.
2. From them, one derives the rotation matrix, $\mathbf{R}^{\mathrm{T}}$, strip by strip from top to bottom as first row: p_0; for the next three rows =MMULT(MMULT(MSQRT(MMULT(TRANSPOSE(L_1),L_1)),MINVERSE(L_1)),P_1); for the second set of three rows; =MMULT(MMULT(MSQRT(MMULT(TRANSPOSE(L_12),L_12)), MINVERSE(L_12)),P_12) for the final set of three rows; and =MMULT(MMULT(MSQRT(MMULT(TRANSPOSE(L_11),L_11)),MINVERSE(L_11)),P_11); thus building $\mathbf{R}^{\mathrm{T}}$ in Cells N25:X36.
3. $\mathbf{W} = \mathbf{XR}^{-\mathrm{T}}$ derives $\mathbf{W}$ in A19:K36. Cells N49:X60 show $\mathbf{W}^{\mathbf{T}}\mathbf{W} = \mathbf{I}$ , proving that $\mathbf{W}$ is is orthogonal in the strictest sense.
4. From inspection, $\mathbf{W} = \mathbf{XR}^{-\mathrm{T}}$, $\mathbf{U} = \mathbf{XS}^{-1}$, and $\mathbf{V} = \mathbf{XL}^{-\mathrm{T}}$ differ from one another and from $\mathbf{R}^{\mathrm{T}}$ (compare Cells A19:K35, N13:X24, N25:X36, and N37:X48). Yet $\mathbf{W}^{\mathbf{T}}\mathbf{W} = \mathbf{U}^{\mathbf{T}}\mathbf{U} = \mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{I}$; thus, all are strictly orthogonal‡.
5. All these matrices generate identical predictions with their respective coefficients; that is, $\hat{\mathbf{y}} = \mathbf{Xa} = \mathbf{Ub} = \mathbf{Wc} = \mathbf{Vd}$, as show in Columns L:M of the spreadsheet (with L3:L18 holding the original y values which form the basis of all the regressions). One derives the respective coefficients directly from their matrices ($\mathbf{b} = \mathbf{U}^{\mathrm{T}}\mathbf{y}$, $\mathbf{c} = \mathbf{W}^{\mathrm{T}}\mathbf{y}$, $\mathbf{d} = \mathbf{V}^{\mathrm{T}}\mathbf{y}$) or from the $\mathbf{a}$ coefficients; i.e., $\mathbf{b} = \mathbf{Sa}$, $\mathbf{c} = \mathbf{R}^{\mathrm{T}}\mathbf{a}$, $\mathbf{d} = \mathbf{L}^{\mathrm{T}}\mathbf{a}$, as shown in Columns Z:AA.
6. Columns AB:AH hold the ANOVAs. The ANOVA for the individual components of each block is in AB1:AH15. Note that the sum of squares for each component is simply that coefficient squared. Dividing each sum of squares by the total sum of squares gives individual coefficients of determination (AH4:AH12), each being a direct assessment of practical significance for the respective effect. Cells AB16:AH24 hold the ANOVA for the grouped components. Note that the sum of squares for each group is identical to the sum of squares of its components. Of necessity, both ANOVAs for the components and groups give exactly the same apportionments of variance for SSM, SSR, and SST. Per their respective F tests, all groups are significant at the 95% confidence level. However, the component-wise ANOVA shows that the $w_{22}$ component in the $W_{11}$ group is not statistically significant. It also has virtually no practical significance; therefore, omitting it from the model would be inconsequential.

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† The Chol() and MSQRT() commands require the Real-Statistics add-in for Excel, last accessed 22 November 2026: https://real-statistics.com/free-download/.

‡ One could also show that $\mathbf{RR}^{\mathrm{T}} = \mathbf{LL}^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}}\mathbf{X}$; thus, $\mathbf{RR}^{\mathrm{T}}$ is an alternate (blockwise) decomposition to the Cholesky.

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1.) X Matrix | | | | | | | | | | y | ŷ = | | 2.) $X^TX$ Matrix | | | | | | | | | |
| 1 | Pt | 0 | 1 | 2 | 3 | 12 | 13 | 23 | 11 | 22 | 33 | | Xa | Pt | 0 | 1 | 2 | 3 | 12 | 13 | 23 | 11 | 22 | 33 |
| 2 | 1 | 1 | -0.761 | -0.781 | 0.578 | 0.594 | -0.440 | -0.452 | 0.579 | 0.611 | 0.334 | -0.121 | -0.169 | 0 | 16.000 | 1.441 | -1.721 | 1.942 | 0.196 | 2.359 | -0.889 | 4.333 | 6.024 | 7.414 |
| 3 | 2 | 1 | -0.479 | 0.162 | -0.951 | -0.078 | 0.456 | -0.154 | 0.230 | 0.026 | 0.905 | -0.550 | -0.398 | 1 | 1.441 | 4.333 | 0.196 | 2.359 | -0.871 | 1.623 | -0.317 | 1.484 | 0.403 | 0.339 |
| 4 | 3 | 1 | -0.415 | 0.805 | -0.824 | -0.334 | 0.342 | -0.663 | 0.172 | 0.648 | 0.680 | -1.442 | -1.564 | 2 | -1.721 | 0.196 | 6.024 | -0.889 | 0.403 | -0.317 | -0.110 | -0.871 | -1.592 | -1.305 |
| 5 | 4 | 1 | -0.394 | -0.321 | 0.878 | 0.126 | -0.346 | -0.281 | 0.155 | 0.103 | 0.771 | -1.182 | -1.077 | 3 | 1.942 | 2.359 | -0.889 | 7.414 | -0.317 | 0.339 | -1.305 | 1.623 | -0.110 | 0.560 |
| 6 | 5 | 1 | -0.295 | -0.999 | -0.759 | 0.295 | 0.224 | 0.758 | 0.087 | 0.998 | 0.576 | 1.516 | 1.463 | 12 | 0.196 | -0.871 | 0.403 | -0.317 | 1.699 | -1.065 | 0.855 | -0.118 | 0.264 | -0.345 |
| 7 | 6 | 1 | -0.291 | 0.083 | 0.357 | -0.024 | -0.104 | 0.030 | 0.085 | 0.007 | 0.128 | -0.420 | -0.536 | 13 | 2.359 | 1.623 | -0.317 | 0.339 | -1.065 | 2.043 | -0.345 | 1.178 | 0.855 | 1.626 |
| 8 | 7 | 1 | -0.232 | -0.684 | -0.669 | 0.159 | 0.156 | 0.458 | 0.054 | 0.468 | 0.448 | 1.077 | 1.087 | 23 | -0.889 | -0.317 | -0.110 | -1.305 | 0.855 | -0.345 | 2.493 | -1.065 | -0.592 | -0.494 |
| 9 | 8 | 1 | -0.109 | 0.625 | 0.153 | -0.068 | -0.017 | 0.095 | 0.012 | 0.391 | 0.023 | -1.106 | -0.937 | 11 | 4.333 | 1.484 | -0.871 | 1.623 | -0.118 | 1.178 | -1.065 | 2.257 | 1.699 | 2.043 |
| 10 | 9 | 1 | 0.219 | 0.235 | 0.594 | 0.051 | 0.130 | 0.139 | 0.048 | 0.055 | 0.353 | -0.951 | -1.126 | 22 | 6.024 | 0.403 | -1.592 | -0.110 | 0.264 | 0.855 | -0.592 | 1.699 | 4.096 | 2.493 |
| 11 | 10 | 1 | 0.273 | -0.316 | -0.751 | -0.086 | -0.205 | 0.237 | 0.074 | 0.100 | 0.563 | 0.351 | 0.318 | 33 | 7.414 | 0.339 | -1.305 | 0.560 | -0.345 | 1.626 | -0.494 | 2.043 | 2.493 | 4.600 |
| 12 | 11 | 1 | 0.350 | 0.422 | 0.914 | 0.147 | 0.320 | 0.386 | 0.122 | 0.178 | 0.836 | -1.967 | -1.954 | 3.) $L^T$ Matrix | | | | | | | | | | |
| 13 | 12 | 1 | 0.423 | -0.990 | 0.537 | -0.419 | 0.227 | -0.532 | 0.179 | 0.981 | 0.289 | -1.220 | -1.089 | | 0 | 1 | 2 | 3 | 12 | 13 | 23 | 11 | 22 | 33 |
| 14 | 13 | 1 | 0.665 | 0.907 | -0.177 | 0.603 | -0.118 | -0.161 | 0.442 | 0.823 | 0.031 | -1.286 | -1.303 | 0 | 4.000 | 0.360 | -0.430 | 0.486 | 0.049 | 0.590 | -0.222 | 1.083 | 1.506 | 1.854 |
| 15 | 14 | 1 | 0.715 | -0.362 | 0.700 | -0.259 | 0.501 | -0.254 | 0.512 | 0.131 | 0.491 | -1.917 | -2.059 | 1 | | 2.050 | 0.171 | 1.066 | -0.434 | 0.688 | -0.116 | 0.534 | -0.068 | -0.160 |
| 16 | 15 | 1 | 0.813 | 0.181 | 0.507 | 0.148 | 0.412 | 0.092 | 0.661 | 0.033 | 0.257 | -2.042 | -1.888 | 2 | | | 2.410 | -0.358 | 0.207 | -0.075 | -0.077 | -0.206 | -0.387 | -0.199 |
| 17 | 16 | 1 | 0.960 | -0.687 | 0.854 | -0.660 | 0.820 | -0.587 | 0.922 | 0.473 | 0.729 | -3.323 | -3.351 | 3 | | | | 2.432 | 0.080 | -0.291 | -0.453 | 0.187 | -0.373 | -0.099 |
| 18 | 5.) $W = XR^{-T}$ | | | | | | | | | | | y | ŷ = | 12 | | | | | 1.208 | -0.626 | 0.718 | 0.073 | 0.224 | -0.377 |
| 19 | Pt | 0 | 1 | 2 | 3 | 12 | 13 | 23 | 11 | 22 | 33 | | Wc | 13 | | | | | | 0.860 | 0.208 | 0.298 | 0.019 | 0.422 |
| 20 | 1 | 0.250 | -0.474 | -0.246 | 0.246 | 0.455 | -0.086 | -0.323 | 0.236 | -0.036 | -0.086 | -0.121 | -0.169 | 23 | | | | | | | 1.288 | -0.627 | -0.488 | 0.018 |
| 21 | 2 | 0.250 | -0.203 | 0.098 | -0.365 | -0.020 | 0.404 | -0.154 | -0.151 | -0.438 | 0.115 | -0.550 | -0.398 | 11 | | | | | | | | 0.484 | -0.483 | 0.024 |
| 22 | 3 | 0.250 | -0.199 | 0.368 | -0.303 | -0.278 | 0.205 | -0.415 | 0.011 | 0.259 | 0.108 | -1.442 | -1.564 | 22 | | | | | | | | | 1.006 | -0.324 |
| 23 | 4 | 0.250 | -0.322 | -0.055 | 0.343 | -0.068 | -0.245 | -0.047 | -0.044 | -0.088 | 0.451 | -1.182 | -1.077 | 33 | | | | | | | | | | 0.814 |
| 24 | 5 | 0.250 | -0.094 | -0.386 | -0.338 | 0.215 | 0.181 | 0.382 | 0.111 | 0.412 | 0.027 | 1.516 | 1.463 | 3.) $R^T$ Matrix | | | | | | | | | | |
| 25 | 6 | 0.250 | -0.228 | 0.096 | 0.136 | -0.223 | -0.128 | 0.154 | 0.131 | -0.170 | -0.395 | -0.420 | -0.536 | | 0 | 1 | 2 | 3 | 12 | 13 | 23 | 11 | 22 | 33 |
| 26 | 7 | 0.250 | -0.078 | -0.254 | -0.300 | 0.087 | 0.059 | 0.222 | -0.148 | -0.097 | -0.157 | 1.077 | 1.087 | 0 | 4.000 | 0.360 | -0.430 | 0.486 | 0.049 | 0.590 | -0.222 | 1.083 | 1.506 | 1.854 |
| 27 | 8 | 0.250 | -0.127 | 0.312 | 0.052 | -0.276 | -0.127 | 0.201 | 0.249 | 0.287 | -0.379 | -1.106 | -0.937 | 1 | | 1.992 | 0.095 | 0.475 | -0.446 | 0.739 | -0.004 | 0.481 | 0.033 | -0.127 |
| 28 | 9 | 0.250 | 0.013 | 0.153 | 0.185 | -0.042 | 0.003 | 0.215 | -0.343 | -0.149 | -0.104 | -0.951 | -1.126 | 2 | | 0.095 | 2.410 | -0.144 | 0.191 | -0.063 | -0.112 | -0.168 | -0.414 | -0.212 |
| 29 | 10 | 0.250 | 0.186 | -0.116 | -0.371 | -0.229 | -0.611 | 0.101 | 0.193 | -0.246 | 0.321 | 0.351 | 0.318 | 3 | | 0.475 | -0.144 | 2.633 | -0.039 | -0.117 | -0.460 | 0.321 | -0.348 | -0.118 |
| 30 | 11 | 0.250 | 0.046 | 0.236 | 0.306 | 0.060 | 0.235 | 0.416 | -0.124 | 0.201 | 0.472 | -1.967 | -1.954 | 12 | | | | | 1.109 | -0.348 | 0.330 | 0.332 | 0.356 | -0.241 |
| 31 | 12 | 0.250 | 0.158 | -0.366 | 0.109 | -0.219 | -0.141 | -0.278 | -0.560 | 0.303 | -0.189 | -1.220 | -1.089 | 13 | | | | | -0.348 | 1.003 | -0.064 | 0.291 | -0.026 | 0.512 |
| 32 | 13 | 0.250 | 0.304 | 0.400 | -0.146 | 0.553 | -0.303 | -0.226 | -0.099 | 0.218 | -0.010 | -1.286 | -1.303 | 23 | | | | | 0.330 | -0.064 | 1.451 | -0.541 | -0.402 | -0.033 |
| 33 | 14 | 0.250 | 0.280 | -0.107 | 0.164 | -0.041 | 0.139 | -0.077 | -0.161 | -0.269 | -0.085 | -1.917 | -2.059 | 11 | | | | | | | | 0.460 | -0.152 | -0.011 |
| 34 | 15 | 0.250 | 0.336 | 0.112 | 0.092 | 0.255 | 0.129 | 0.088 | 0.206 | -0.306 | -0.211 | -2.042 | -1.888 | 22 | | | | | | | | -0.152 | 1.092 | -0.174 |
| 35 | 16 | 0.250 | 0.403 | -0.245 | 0.192 | -0.229 | 0.285 | -0.259 | 0.493 | 0.118 | 0.122 | -3.323 | -3.351 | 33 | | | | | | | | -0.011 | -0.174 | 0.859 |

**FIGURE 11.8-5** Part 1: the spreadsheet. This section of the spreadsheet shows 1.) The starting X matrix. 2.) The subsequent $X^TX$ Matrix. The transposed Cholesky Matrix. 4.) The rotation matrix, R. The W Matrix.

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 36 | **7.) U = XS$^{-1}$** | | | | | | | | | | | **y** | **ŷ =** | **6.) S = √(X$^T$X)** | | | | | | | | | | |
| 37 | **Pt** | **0** | **1** | **2** | **3** | **12** | **13** | **23** | **11** | **22** | **33** | | **Ub** | | **0** | **1** | **2** | **3** | **12** | **13** | **23** | **11** | **22** | **33** |
| 38 | **1** | 0.164 | -0.431 | -0.267 | 0.199 | 0.390 | -0.238 | -0.247 | 0.419 | 0.147 | -0.023 | -0.121 | -0.169 | **0** | 3.517 | 0.172 | -0.151 | 0.280 | 0.076 | 0.308 | -0.059 | 0.735 | 1.045 | 1.327 |
| 39 | **2** | 0.246 | -0.228 | 0.042 | -0.385 | 0.134 | 0.385 | -0.216 | 0.015 | -0.284 | 0.326 | -0.550 | -0.398 | **1** | 0.172 | 1.903 | 0.102 | 0.494 | -0.252 | 0.451 | 0.030 | 0.388 | 0.066 | -0.065 |
| 40 | **3** | 0.135 | -0.247 | 0.407 | -0.285 | -0.273 | 0.130 | -0.355 | 0.028 | 0.309 | 0.246 | -1.442 | -1.564 | **2** | -0.151 | 0.102 | 2.392 | -0.182 | 0.144 | -0.019 | -0.098 | -0.184 | -0.346 | -0.232 |
| 41 | **4** | 0.158 | -0.167 | -0.075 | 0.327 | 0.041 | -0.392 | -0.143 | -0.080 | -0.074 | 0.462 | -1.182 | -1.077 | **3** | 0.280 | 0.494 | -0.182 | 2.617 | -0.026 | -0.019 | -0.300 | 0.308 | -0.149 | 0.025 |
| 42 | **5** | 0.074 | -0.144 | -0.315 | -0.212 | 0.094 | 0.139 | 0.529 | 0.069 | 0.487 | 0.150 | 1.516 | 1.463 | **12** | 0.076 | -0.252 | 0.144 | -0.026 | 1.116 | -0.433 | 0.362 | 0.109 | 0.136 | -0.115 |
| 43 | **6** | 0.448 | -0.248 | 0.048 | 0.136 | -0.179 | -0.073 | 0.093 | 0.029 | -0.177 | -0.270 | -0.420 | -0.536 | **13** | 0.308 | 0.451 | -0.019 | -0.019 | -0.433 | 1.136 | -0.009 | 0.309 | 0.131 | 0.392 |
| 44 | **7** | 0.301 | -0.070 | -0.288 | -0.255 | 0.139 | 0.159 | 0.201 | -0.104 | 0.030 | -0.014 | 1.077 | 1.087 | **23** | -0.059 | 0.030 | -0.098 | -0.300 | 0.362 | -0.009 | 1.441 | -0.379 | -0.177 | -0.070 |
| 45 | **8** | 0.355 | -0.190 | 0.336 | 0.111 | -0.343 | -0.096 | 0.241 | 0.050 | 0.206 | -0.286 | -1.106 | -0.937 | **11** | 0.735 | 0.388 | -0.184 | 0.308 | 0.109 | 0.309 | -0.379 | 1.008 | 0.255 | 0.326 |
| 46 | **9** | 0.353 | 0.068 | 0.089 | 0.217 | 0.114 | 0.135 | 0.041 | -0.292 | -0.127 | -0.001 | -0.951 | -1.126 | **22** | 1.045 | 0.066 | -0.346 | -0.149 | 0.136 | 0.131 | -0.177 | 0.255 | 1.626 | 0.283 |
| 47 | **10** | 0.298 | 0.340 | -0.142 | -0.401 | -0.229 | -0.571 | 0.129 | 0.031 | -0.183 | 0.267 | 0.351 | 0.318 | **33** | 1.327 | -0.065 | -0.232 | 0.025 | -0.115 | 0.392 | -0.070 | 0.326 | 0.283 | 1.557 |
| 48 | **11** | 0.044 | 0.080 | 0.273 | 0.409 | 0.118 | 0.138 | 0.345 | -0.146 | 0.120 | 0.535 | -1.967 | -1.954 | **9.) W$^T$W = R$^{-T}$(X$^T$X)R$^{-1}$ = I** | | | | | | | | | | |
| 49 | **12** | 0.239 | 0.218 | -0.386 | 0.141 | -0.150 | 0.088 | -0.387 | -0.422 | 0.420 | -0.107 | -1.220 | -1.089 | | **0** | **1** | **2** | **3** | **12** | **13** | **23** | **11** | **22** | **33** |
| 50 | **13** | 0.182 | 0.417 | 0.384 | -0.147 | 0.480 | -0.186 | -0.160 | 0.133 | 0.391 | -0.082 | -1.286 | -1.303 | **0** | 1 | | | | | | | | | |
| 51 | **14** | 0.258 | 0.263 | -0.159 | 0.150 | 0.011 | 0.270 | -0.162 | 0.032 | -0.166 | 0.030 | -1.917 | -2.059 | **1** | | 1 | | | | | | | | |
| 52 | **15** | 0.274 | 0.273 | 0.071 | 0.078 | 0.188 | 0.217 | 0.116 | 0.390 | -0.204 | -0.127 | -2.042 | -1.888 | **2** | | | 1 | | | | | | | |
| 53 | **16** | -0.010 | 0.240 | -0.170 | 0.195 | -0.457 | 0.203 | -0.085 | 0.585 | 0.151 | 0.220 | -3.323 | -3.351 | **3** | | | | 1 | | | | | | |
| 54 | **8.) V= XL$^{-T}$** | | | | | | | | | | | **y** | **ŷ =** | **12** | | | | | 1 | | | | | |
| 55 | **Pt** | **0** | **1** | **2** | **3** | **12** | **13** | **23** | **11** | **22** | **33** | | **Vd** | **13** | | | | | | 1 | | | | |
| 56 | **1** | 0.250 | -0.415 | -0.250 | 0.333 | 0.354 | -0.002 | -0.440 | 0.237 | 0.054 | -0.072 | -0.121 | -0.169 | **23** | | | | | | | 1 | | | |
| 57 | **2** | 0.250 | -0.278 | 0.132 | -0.300 | -0.177 | 0.362 | -0.159 | -0.009 | -0.476 | 0.027 | -0.550 | -0.398 | **11** | | | | | | | | 1 | | |
| 58 | **3** | 0.250 | -0.246 | 0.396 | -0.223 | -0.428 | 0.071 | -0.321 | -0.073 | 0.226 | 0.151 | -1.442 | -1.564 | **22** | | | | | | | | | 1 | |
| 59 | **4** | 0.250 | -0.236 | -0.072 | 0.404 | -0.005 | -0.258 | -0.015 | -0.025 | -0.179 | 0.425 | -1.182 | -1.077 | **33** | | | | | | | | | | 1 |
| 60 | **5** | 0.250 | -0.188 | -0.356 | -0.332 | 0.249 | 0.277 | 0.293 | -0.025 | 0.414 | 0.105 | 1.516 | 1.463 | **10.) U$^T$U = I** | | | | | | | | | | |
| 61 | **6** | 0.250 | -0.186 | 0.092 | 0.192 | -0.125 | -0.162 | 0.219 | 0.187 | -0.046 | -0.406 | -0.420 | -0.536 | | **0** | **1** | **2** | **3** | **12** | **13** | **23** | **11** | **22** | **33** |
| 62 | **7** | 0.250 | -0.157 | -0.228 | -0.290 | 0.123 | 0.107 | 0.183 | -0.107 | -0.107 | -0.183 | 1.077 | 1.087 | **0** | 1 | | | | | | | | | |
| 63 | **8** | 0.250 | -0.097 | 0.311 | 0.101 | -0.161 | -0.169 | 0.280 | 0.156 | 0.414 | -0.304 | -1.106 | -0.937 | **1** | | 1 | | | | | | | | |
| 64 | **9** | 0.250 | 0.063 | 0.138 | 0.187 | 0.019 | 0.019 | 0.217 | -0.276 | -0.224 | -0.154 | -0.951 | -1.126 | **2** | | | 1 | | | | | | | |
| 65 | **10** | 0.250 | 0.089 | -0.093 | -0.411 | -0.006 | -0.633 | 0.191 | 0.253 | -0.230 | 0.290 | 0.351 | 0.318 | **3** | | | | 1 | | | | | | |
| 66 | **11** | 0.250 | 0.127 | 0.211 | 0.301 | 0.101 | 0.293 | 0.369 | -0.192 | 0.063 | 0.488 | -1.967 | -1.954 | **12** | | | | | 1 | | | | | |
| 67 | **12** | 0.250 | 0.162 | -0.378 | 0.044 | -0.237 | -0.228 | -0.194 | -0.622 | 0.147 | -0.180 | -1.220 | -1.089 | **13** | | | | | | 1 | | | | |
| 68 | **13** | 0.250 | 0.280 | 0.401 | -0.187 | 0.533 | -0.172 | -0.367 | -0.163 | 0.176 | 0.019 | -1.286 | -1.303 | **23** | | | | | | | 1 | | | |
| 69 | **14** | 0.250 | 0.305 | -0.127 | 0.086 | -0.099 | 0.113 | -0.067 | -0.067 | -0.285 | -0.142 | -1.917 | -2.059 | **11** | | | | | | | | 1 | | |
| 70 | **15** | 0.250 | 0.353 | 0.095 | 0.018 | 0.221 | 0.201 | 0.003 | 0.297 | -0.184 | -0.242 | -2.042 | -1.888 | **22** | | | | | | | | | 1 | |
| 71 | **16** | 0.250 | 0.424 | -0.271 | 0.076 | -0.363 | 0.180 | -0.191 | 0.428 | 0.238 | 0.179 | -3.323 | -3.351 | **33** | | | | | | | | | | 1 |

**FIGURE 11.8-5** Part 2: the spreadsheet (continued). This section of the spreadsheet shows 6.) S, the square root of the $X^TX$ matrix. 7.) The resulting $U = XS^{-1}$ matrix. 8.) V, the Cholesky transform of the X matrix. 9.) Proof that the blockwise W matrix is orthogonal. 10. Proof that the U matrix is orthogonal.

| | Y | Z | AA | AB | AC | AD | AE | AF | AG | AH | AI |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Coeff | | Term | ANOVA, Component Model | | | | | | |
| 1 | Pt | a | | | SS | DF | MS | F | p | $R_a^2$ | stat sig |
| 2 | 0 | -0.051 | | $w_0$ | 13.294 | — | — | — | — | — | — |
| 3 | 1 | -0.365 | | $w_1$ | 5.458 | 1 | 5.458 | 172.24 | 0.0000 | 24.5% | * |
| 4 | 2 | -0.996 | | $w_2$ | 2.763 | 1 | 2.763 | 87.20 | 0.0001 | 12.4% | * |
| 5 | 3 | -0.865 | | $w_3$ | 7.512 | 1 | 7.512 | 237.06 | 0.0000 | 33.7% | * |
| 6 | 12 | 1.016 | | $w_{12}$ | 0.941 | 1 | 0.941 | 29.68 | 0.0016 | 4.2% | * |
| 7 | 13 | -0.160 | | $w_{13}$ | 1.644 | 1 | 1.644 | 51.89 | 0.0004 | 7.4% | * |
| 8 | 23 | 0.415 | | $w_{23}$ | 3.163 | 1 | 3.163 | 99.83 | 0.0001 | 14.2% | * |
| 9 | 11 | -1.278 | | $w_{11}$ | 0.286 | 1 | 0.286 | 9.04 | 0.0238 | 1.3% | * |
| 10 | 22 | -0.288 | | $w_{22}$ | 0.000 | 1 | 0.000 | 0.00 | 0.9657 | 0.0% | |
| 11 | 33 | -0.737 | | $w_{33}$ | 0.323 | 1 | 0.323 | 10.21 | 0.0187 | 1.5% | * |
| 12 | | Coeff | | M | 22.091 | 9 | 2.455 | 77.46 | 0.0000 | | |
| 13 | Pt | $d=L^Ta=V^Ty$ | | R | 0.190 | 6 | 0.032 | s = | 0.178 | | |
| 14 | 0 | -3.646 | -3.646 | T | 22.281 | 15 | | $R_2$ = | 99.15% | 99.15% | |
| 15 | 1 | -2.983 | -2.983 | Term | ANOVA, Group Model | | | | | | |
| 16 | 2 | -1.379 | -1.379 | | SS | DF | MS | F | p | $R_a^2$ | $e_j=\Sigma b_{jk}$ |
| 17 | 3 | -2.221 | -2.221 | $w_0$ | 1.159 | — | — | — | — | — | -3.646 |
| 18 | 12 | 1.746 | 1.746 | $W_1$ | 15.733 | 3 | 5.244 | 165.50 | 0.0000 | 70.6% | 3.966 |
| 19 | 13 | -0.749 | -0.749 | $W_{12}$ | 5.748 | 3 | 1.916 | 60.47 | 0.0001 | 25.8% | 2.398 |
| 20 | 23 | 1.463 | 1.463 | $W_{11}$ | 0.610 | 3 | 0.203 | 6.42 | 0.0266 | 2.7% | 0.781 |
| 21 | 11 | -0.497 | -0.497 | M | 22.091 | 9 | 2.455 | 77.46 | 0.0000 | | |
| 22 | 22 | -0.051 | -0.051 | R | 0.190 | 6 | 0.032 | s = | 0.178 | | |
| 23 | 33 | -0.600 | -0.600 | T | 22.281 | 15 | | $R_2$ = | 99.15% | 99.15% | |
| 24 | | Coeff | | Group Model, G | | | | | | | |
| 25 | Pt | $c=R^Ta=W^Ty$ | | Pt | $g_0$ | $g_1$ | $g_{12}$ | $g_{11}$ | y | $\hat{y}=Ge$ | |
| 26 | 0 | -3.646 | -3.646 | 1 | 0.250 | 0.213 | -0.010 | -0.099 | -0.121 | -0.169 | |
| 27 | 1 | -2.336 | -2.336 | 2 | 0.250 | 0.331 | -0.339 | 0.016 | -0.550 | -0.398 | |
| 28 | 2 | -1.662 | -1.662 | 3 | 0.250 | 0.172 | -0.530 | -0.084 | -1.442 | -1.564 | |
| 29 | 3 | -2.741 | -2.741 | 4 | 0.250 | -0.024 | 0.068 | -0.299 | -1.182 | -1.077 | |
| 30 | 12 | 0.970 | 0.970 | 5 | 0.250 | 0.451 | 0.274 | -0.092 | 1.516 | 1.463 | |
| 31 | 13 | -1.282 | -1.282 | 6 | 0.250 | 0.000 | 0.093 | 0.196 | -0.420 | -0.536 | |
| 32 | 23 | 1.779 | 1.779 | 7 | 0.250 | 0.360 | 0.168 | 0.215 | 1.077 | 1.087 | |
| 33 | 11 | -0.535 | -0.535 | 8 | 0.250 | -0.092 | 0.106 | 0.108 | -1.106 | -0.937 | |
| 34 | 22 | 0.008 | 0.008 | 9 | 0.250 | -0.200 | 0.141 | 0.309 | -0.951 | -1.126 | |
| 35 | 33 | -0.569 | -0.569 | 10 | 0.250 | 0.196 | 0.309 | -0.369 | 0.351 | 0.318 | |
| 36 | Y | Z | AA | 11 | 0.250 | -0.337 | 0.207 | -0.257 | -1.967 | -1.954 | |
| 37 | | Coeff | | 12 | 0.250 | -0.015 | -0.219 | 0.525 | -1.220 | -1.089 | |
| 38 | Pt | $b=Sa=U^Ty$ | | 13 | 0.250 | -0.246 | 0.218 | 0.077 | -1.286 | -1.303 | |
| 39 | 0 | -2.547 | -2.547 | 14 | 0.250 | -0.233 | -0.148 | 0.170 | -1.917 | -2.059 | |
| 40 | 1 | -2.013 | -2.013 | 15 | 0.250 | -0.308 | 0.100 | 0.009 | -2.042 | -1.888 | |
| 41 | 2 | -1.639 | -1.639 | 16 | 0.250 | -0.267 | -0.437 | -0.425 | -3.323 | -3.351 | |
| 42 | 3 | -2.794 | -2.794 | | | | | | | | |
| 43 | 12 | 1.228 | 1.228 | | | | | | | | |
| 44 | 13 | -1.491 | -1.491 | | | | | | | | |
| 45 | 23 | 1.903 | 1.903 | | | | | | | | |
| 46 | 11 | -1.958 | -1.958 | | | | | | | | |
| 47 | 22 | -0.564 | -0.564 | | | | | | | | |
| 48 | 33 | -1.687 | -1.687 | | | | | | | | |

| 49 | 11.) $V^TV = I$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | | 0 | 1 | 2 | 3 | 12 | 13 | 23 | 11 | 22 | 33 |
| 51 | 0 | 1 | | | | | | | | | |
| 52 | 1 | | 1 | | | | | | | | |
| 53 | 2 | | | 1 | | | | | | | |
| 54 | 3 | | | | 1 | | | | | | |
| 55 | 12 | | | | | 1 | | | | | |
| 56 | 13 | | | | | | 1 | | | | |
| 57 | 23 | | | | | | | 1 | | | |
| 58 | 11 | | | | | | | | 1 | | |
| 59 | 22 | | | | | | | | | 1 | |
| 60 | 33 | | | | | | | | | | 1 |

**FIGURE 11.8-5** Part 3: the spreadsheet continued. Columns Z:AA give the coefficients of the various models with the blockwise coefficients in Z27:AA36. Y49:AI60 proves that V is strictly orthogonal. AB1:AI14 gives the component wise ANOVA. AB16:AI24 gives the groupwise ANOVA. AB25:AH42 shows the groupwise contributions and regressed y values.