COMPRESSIBLE FLOW EQUATIONS FOR FUEL ORIFICES

Revision 3

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Nomenclature

Variables

Roman

A	cross-sectional area
C_o	orifice coefficient (empirical)
C_p	heat capacity, molar, isobaric
C_V	heat capacity, molar, isometric
H	enthalpy, molar
N_G	mass flow number, dimensionless
P	pressure
Q	heat
$\dot{Q}$	heat release rate (thermal power)
r	pressure ratio
R	gas constant, ideal, universal
S	entropy, molar
T	temperature, absolute
U	internal energy
v	velocity
$\hat{V}$	volume, molar
W	molecular weight

Greek

β	diameter ratio of the orifice diameter to the upstream conduit diameter
γ	heat capacity ratio, C_p/C_V
ρ	density

Operators

General

d	differential (prefix, unary)
δ	inexact differential (prefix, unary)
Δ	difference (prefix, unary)
$=$	equality (infix, binary)
$-$	minus (binary when infix, unary when prefix)
$+$	plus (infix, binary)
$\frac{a}{b}$	division (infix, binary)
ab	proximate entities indicate multiplication

Superscripts

e.g. $^{\frac{2}{\gamma}}$	all alphanumeric superscripts are exponential powers
e.g., $\overline{H}$	an overbar indicates a specific (mass-based) property
e.g., $\hat{V}$	a tilde explicitly declares a molar property (properties are mole-based unless otherwise indicated)

Subscripts (all subscripts are postfix and unary)

none	upstream state
c	at the critical or choked (sonic flow) condition
g	ambient (downstream) state
o	at the orifice
$_1$	property evaluated at state 1
$_2$	property evaluated at state 2

Adiabatic Isentropic Relations

From the first law of thermodynamics $dU = TdS - Pd\hat{V}$, or $(\partial U)_S = -P(\partial \hat{V})_S$; i.e., conservation of energy. But $\left(\frac{\partial U}{\partial T}\right)_{\hat{V}} \equiv C_V$; thus $C_V dT = -Pd\hat{V}$, and from the ideal gas law, $P\hat{V} = RT$, leading to $C_V dT = -RT\frac{d\hat{V}}{\hat{V}}$. Expressing the relation in terms of γ (where $\gamma = \frac{C_p}{C_v}$), noting that $R = C_p - C_V$, and rearranging, gives a relation that may be integrated at once: $\int_{T_1}^{T_2}\frac{dT}{T} = -(\gamma - 1)\int_{\tilde{V}_1}^{\tilde{V}_2}\frac{d\hat{V}}{\hat{V}}$, leading to $\frac{T_2}{T_1} = \left(\frac{\hat{V}_1}{\hat{V}_2}\right)^{\gamma-1}$. Making repeated use of the ideal gas law, this may be expressed in the following series.

$$\frac{P_2}{P_1} = \left(\frac{\hat{V}_1}{\hat{V}_2}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

energy balance (adiabatic, isentropic, ideal gas) (1)

Or equivalently, $\partial\left(P\hat{V}^{\gamma}\right)_{H,S} = \partial\left(\frac{P^{\frac{\gamma-1}{\gamma}}}{T}\right)_{H,S} = \partial\left(\frac{P}{\rho^{\gamma}}\right)_{H,S} = \partial\left(T\hat{V}^{\gamma-1}\right)_{H,S} = \partial\left(\frac{T}{\rho^{\gamma-1}}\right)_{H,S} = 0$.

Speed of Sound

We shall find that the maximum free propagation velocity for a small disturbance has an upper bound referred to as the *sonic velocity* or the *speed of sound*. For the sake of developing the system boundary and governing equations, consider the system to be a frictionless constant-area duct where a one-dimensional infinitesimal disturbance propagates without loss of mass or momentum: $\partial(\rho A v_c)_S = 0$, $d\left(pA + \rho A v_c^2\right) = 0$, respectively. The mass balance may be expanded and simplified to give $\frac{d\rho}{\rho} = -\frac{dv_c}{v_c}$. From the momentum balance we obtain $\frac{dP}{\rho} = -v_c dv_c$. Combining these equations to eliminate dv_c gives $\frac{dP}{d\rho} = v_c^2$. Since we derived this equation under adiabatic and reversible (isentropic) conditions, it is sometimes explicitly declared as $\left(\frac{\partial P}{\partial \rho}\right)_{H,S} = v_c^2$.

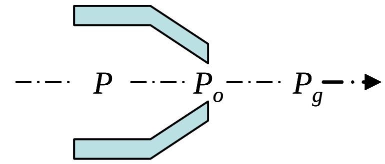
Now, from Equation (1) a differential energy balance on an ideal gas undergoing an adiabatic expansion is $d\left(\frac{P}{\rho^{\gamma}}\right) = 0$, which expands and simplifies to $\frac{dP}{d\rho} = \gamma\frac{P}{\rho}$; using $\frac{dP}{d\rho} = v_c^2$ to eliminate $\frac{dP}{d\rho}$ and taking the square root gives

$$v_c = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{W}}$$

speed of sound (2)

the latter quantity being derived from the ideal gas law for P/ρ. This is the fastest possible rate of propagation for a small pressure disturbance in a constant area duct and is known as the *critical velocity*, *sonic velocity*, or *speed of sound*.

Critical Pressure Ratio for an Orifice



Consider the nozzle at right having pressures as shown and converging to an orifice. For this case, an energy balance must consider internal energy, pressure-volume work, and kinetic energy: $\Delta\left(U + PV + \frac{1}{2}mv^2\right) = 0$*. By definition, $U + PV = H$. With this substitution and dividing by mass to obtain intensive properties, we derive $\Delta\left(\overline{H} + \frac{v^2}{2}\right) = 0$. For the orifice and ambient states, conservation of energy gives $\overline{H}_o + \frac{v_o^2}{2} = \overline{H} + \frac{v^2}{2}$. If the orifice is much smaller than the duct, then $v_o >> v$, and v may be neglected. Then solving for v_o with the substitutions $\overline{H} = \frac{C_p}{W}T$ and $C_p = \left(\frac{\gamma}{\gamma - 1}\right)R$ gives $v_o^2 = \left(\frac{2}{\gamma-1}\right)\left(\frac{\gamma R T_o}{W}\right)\left(\frac{T}{T_o} - 1\right)$. But $\frac{\gamma R T_o}{W} = v_c^2$ and the equation may be recast in terms of the Mach number at the orifice, $N_M = \frac{v_o}{v_c}$: $\frac{T}{T_o} = 1 + \left(\frac{\gamma - 1}{2}\right)N_M^2$. Solving for T/T_o with N_M = 1 gives the critical temperature ratio: $\frac{T}{T_o} = \frac{\gamma+1}{2}$. Using Equation (1) to recast in this in terms of pressure, we obtain

$$r_c = \frac{P}{P_c} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$ critical pressure (adiabatic, isentropic, ideal gas) (3)

Compressible Flow Through an Orifice

Starting again from the conservation of energy, we have $\overline{H}_o + \frac{v_o^2}{2} = \overline{H} + \frac{v^2}{2}$ or $\frac{v^2}{2}\left[\left(\frac{v_o}{v}\right)^2 - 1\right] = \overline{H} - \overline{H}_o$.

Substituting for enthalpy gives $\left(\frac{\gamma}{\gamma-1}\right)\left[\frac{P}{\rho} - \frac{P_O}{\rho_o}\right] = \frac{v^2}{2}\left[\left(\frac{v_o}{v}\right)^2 - 1\right]$. Noting that $\frac{P}{\rho} = \frac{RT}{W}$ from the ideal gas law and factoring it from the previous equation gives $\frac{RT}{W}\left(\frac{\gamma}{\gamma-1}\right)\left(1 - \frac{\rho/\rho_o}{P/P_O}\right) = \frac{v^2}{2}\left[\left(\frac{v_o}{v}\right)^2 - 1\right]$. However, from Equation (1) $\frac{\rho}{\rho_o} = \left(\frac{P}{P_o}\right)^{\frac{1}{\gamma}}$, leading to $\frac{RT}{W}\left(\frac{\gamma}{\gamma-1}\right)\left[1 - \left(\frac{P}{P_o}\right)^{\frac{1-\gamma}{\gamma}}\right] = \frac{v^2}{2}\left[\left(\frac{v_o}{v}\right)^2 - 1\right]$. Additionally, a mass balance –

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* The integrated energy balance between two stations is $\Delta\left(\frac{1}{\gamma-1}\frac{RT}{W} + \frac{P}{\rho} + \frac{v^2}{2}\right) = 0$. Noting from Equation (1) that $\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$, when $P_2/P_1 \approx 1$ (that is, low flow conditions) densities and temperatures may be reasonably well represented by upstream conditions. In such a case, the compressible relation reduces to the incompressible form: $\left(\frac{P_2}{\rho} - \frac{P_1}{\rho}\right) + \left(\frac{v_2^2}{2} - \frac{v_1^2}{2}\right) = 0$.

$\rho A v = \rho_o A_o v_o$ – leads to $\left(\frac{v_o}{v}\right)^2 = \left(\frac{\rho A}{\rho_o A_o}\right)^2 = \left(\frac{P}{P_o}\right)^{\frac{2}{\gamma}} \left(\frac{A}{A_o}\right)^2 = \left(\frac{P}{P_o}\right)^{\frac{2}{\gamma}} \frac{1}{\beta^4}$. With this substitution the pressure-ratio form of the mass balance becomes $\frac{RT}{W}\left(\frac{\gamma}{\gamma-1}\right)\left[1-\left(\frac{P}{P_o}\right)^{\frac{1-\gamma}{\gamma}}\right] = \frac{v^2}{2}\left[\left(\frac{P}{P_o}\right)^{\frac{2}{\gamma}}\frac{1}{\beta^4}-1\right]$. Solving for *v*, multiplying by ρA to give mass flow, and accounting for non idealities with an empirical coefficient, C_o, gives

gives $\dot{m} = C_o \rho A \sqrt{\frac{2RT}{W}\left(\frac{\gamma}{\gamma-1}\right)\left[\frac{1-r_o^{\frac{1-\gamma}{\gamma}}}{r_o^{\frac{2}{\gamma}}\frac{1}{\beta^4}-1}\right]} = C_o A_o P \sqrt{\frac{2W}{RT}\left(\frac{\gamma}{\gamma-1}\right)\left[\frac{1-r_o^{\frac{1-\gamma}{\gamma}}}{r_o^{\frac{2}{\gamma}}-\beta^4}\right]}$. This is an appropriate form of the equation for a performance problem, where the orifice area is known and mass flow is desired as a function of pressure. The equation may also be expressed in terms of heat release, $\dot{Q}$, by noting that $\dot{Q} = \dot{m}\Delta H$. If desired, non-dimensional expression may also be developed using $N_G = \frac{\dot{m}}{A_o P}\sqrt{\frac{RT}{W}} = \frac{\dot{Q}}{A_o P \Delta H}\sqrt{\frac{RT}{W}}$. Now when the flow through the orifice is less than the critical velocity, $r_o = r_g = \frac{P}{P_g}$. However, once the orifice reaches critical velocity, $r_o = r_c = \frac{P}{P_c} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$. Thus, the final equation reduces to

$$\dot{m} = C_o A_o P \sqrt{\frac{2W}{RT}\left(\frac{\gamma}{\gamma-1}\right)\left[\frac{1-r_o^{\frac{1-\gamma}{\gamma}}}{r_o^{\frac{2}{\gamma}}-\beta^4}\right]} \quad \text{where} \quad \begin{matrix} r_o = r_g & if & r_g < r_c \\ r_o = r_c & if & r_g \geq r_c \end{matrix}$$

performance equation (4)

However, in the design problem, the required mass flow is known and the orifice area must be determined. In this case, one must solve for the orifice area in terms of $\beta = \sqrt[4]{\frac{r_o^{\frac{2}{\gamma}}\left(\frac{\gamma-1}{\gamma}\right)K^2}{\frac{2RT}{W} - r_o^{\frac{1-\gamma}{\gamma}} + \left(\frac{\gamma-1}{\gamma}\right)K^2}}$ where $K = \frac{\dot{m}}{C_o AP} = \frac{\dot{Q}}{C_o AP\Delta H}$.

$$\beta = \sqrt[4]{\frac{r_o^{\frac{2}{\gamma}}\left(\frac{\gamma-1}{\gamma}\right)K^2}{\frac{2RT}{W} - r_o^{\frac{1-\gamma}{\gamma}} + \left(\frac{\gamma-1}{\gamma}\right)K^2}} \quad \text{where} \quad \begin{matrix} r_o = r_g & if & r_g < r_c \\ r_o = r_c & & r_g \geq r_c \end{matrix}$$

design equation (5)